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Fluctuation of the Free Energy in the Hopfield Model

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We prove that in the ergodic region $[T > J^2(1+\sqrt{\alpha})]$ the deviation of the total free energy of the Hopfield neural network converges in distribution as $N \to \infty$ to a (shifted) Gaussian variable. Moreover, the free energy per site converges in probability to $\lim(1/N)\ln\langle\langle \mathbb{Z}_N \rangle\rangle$.

KEY WORDS: Hopfield model; statistical mechanics central limit theorem.

1. INTRODUCTION

The fluctuations of physical observables have been studied in many cases in order to have a deeper understanding of the phase diagram of the statistical mechanics system under consideration. In the study of disordered systems the question about fluctuations arises also because of the randomness of the interaction. The first question to investigate concerns the convergence of the free energy to the averaged one w.r.t. the distribution of the random variables J_{ij} when the number of spins goes to infinity (only results of this kind about systems with a two-body interaction have been considered up to now). One of the first answers to this question was given in ref. 3 in the case of random independent J_{ij} with a spatial dependence of the type $|i-j|^{-\gamma d}$, $\gamma > 1/2$; the authors proved the convergence in probability of the free energy to the average value. It is also interesting top study the fluctuations of the Edward–Anderson order parameter q_N as shown in ref. 4. It is proved, in the case of the Sherrington–Kirkpatrick (SK) model that if the quantity $E(q_N - E(q_N))^2 \to 0$ for $N \to \infty$, then the mean value of the free energy converges to that found by Kirkpatrick and Sherrington. Then the breaking of replica symmetry at low temperature which makes this free energy wrong is connected with the non-self-averaging property of q_N and thus with the nontrivial asymptotic probability distribution of the

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q_N. In the case of the Hopfield model the asymptotic form of the free energy has been derived in ref. 2 for p finite (p is the number of patterns) and p going to infinity with N in such a way that $\alpha = p/N$ is constant; in that paper, using the saddle point technique and also replica calculations in the second case, the phase diagram was derived for both systems and the critical α was found as a function of the temperature (α_c is defined as the maximal value of α after which there is no memorization in the network). The convergence for any β with probability one to the averaged value has been shown in the case of finite p in refs 5 and 6 with different techniques. In ref. 1 some rigorous results were proved for the SK model, essentially in the high-temperature region. They proved, among other, weaker results, that in this region the deviation of the total free energy from $\ln\langle\langle \mathbb{Z}_N \rangle\rangle$ converges in distribution, as $N \to \infty$, to a shifted Gaussian variable.

We want to extend their results to the Hopfield model (HM) in the ergodic region. Namely, we consider the Hamiltonian

$$H = -\frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j \tag{1.1}$$

where the spins $\sigma_1, \ldots, \sigma_N$ take values $(-1, +1)$ and the patterns ξ_i^μ are i.i.d. random variables. We allow the ξ_i^μ to run on $\mathbb{R}$ with zero mean and all finite moments with a symmetric distribution also if the HM is usually defined through a symmetric distribution over the values $(-1, +1)$. We shall remark the difference when necessary. Here $p = \alpha N$, where p is the number of patterns $\underline{\xi}^\mu$. It is well known that the ergodic region is defined when $\xi_i^\mu \in \{-1, +1\}$ by

$$\mathbb{T} > 1 + \sqrt{\alpha} \tag{1.2}$$

It is claimed that the model in this region has a trivial behavior (i.e., no retrieval states, no spin-glass phase). Nevertheless, no rigorous result is known, and in a forthcoming paper we hope to use the technique developed here to study in a perturbative way the region arounhd the "corner," namely with $1 - \mathbb{T} < \delta$, $\alpha < \delta$, with δ "small." The difference here from ref. 1 is essentially that the variables $\mathbb{J}_{i,j} = (\alpha N)^{-1/2} \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu$ are defined as a finite sum of independent variables and they are "weakly" dependent.

2. NOTATIONS AND RESULTS

The main object we shall consider is the partition function

$$\mathbb{Z}(\underline{\xi}, \beta, \alpha)_N = \sum_{\underline{\sigma}} \exp[-\beta H(\underline{\sigma}, \underline{\xi}, \alpha)] \tag{2.1}$$

where $\underline{\sigma} := [\sigma_1, ..., \sigma_N]$. We shall denote by $\langle\!\langle \cdot \rangle\!\rangle$ the expectation w.r.t. the distribution of the ξ_i^μ variables. Let us write

$$\mathbb{Z}_N(\underline{\xi}, \beta, \alpha) = 2^N \left[\prod_{\mu} \prod_{i<j} \cosh\left(\frac{\beta}{N} \mathbb{J}_{ij}^\mu\right)\right] \cdot \tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) \tag{2.2}$$

where $\mathbb{J}_{ij}^\mu$ is a short-hand notation for $\xi_i^\mu \xi_j^\mu$ and

$$\tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = 2^{-N} \sum_{\underline{\sigma}} \prod_{\mu} \prod_{i<j} \left[1 + \tanh\left(\frac{\beta}{N} \mathbb{J}_{ij}^\mu\right) \sigma_i \sigma_j \right] \tag{2.3}$$

with $\langle\!\langle \tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle \neq 1$. Let us write $\tilde{\mathbb{Z}}_N$ as

$$\tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = 2^{-N} \sum_{\underline{\sigma}} \prod_{\mu} \sum_{\Gamma} \mathbb{W}_\mu(\Gamma) \sigma_\Gamma \tag{2.4}$$

The second summation in (2.4) is over all semisimple graphs with vertices $(1, ..., N)$, where in a semisimple graph each bond (i, j) can appear only once. Let us denote by $|\Gamma|$ the number of the bonds in the graph Γ. If $|\Gamma| = 0$, let us put $\mathbb{W}_\mu(\Gamma) = 1$; otherwise,

$$\mathbb{W}_\mu(\Gamma) = \prod_{b:\text{ edges of } \Gamma} \tanh\left(\frac{\beta}{N} \mathbb{J}_b^\mu\right) \tag{2.5}$$

and

$$\sigma_\Gamma = \prod_{b:\text{ edges of } \Gamma} \sigma_b \tag{2.6}$$

with

$$\sigma_b = \sigma_i \sigma_j \qquad \text{if} \quad b \equiv \{i, j\} \tag{2.6$'$}$$

Sometimes we shall use the short-hand notation

$$\mathbb{W}_\mu(J_{ij}^\mu) \qquad \text{for} \quad \tanh\left(\frac{\beta}{N} \xi_i^\mu \xi_j^\mu\right) \tag{2.5$'$}$$

Now let us observe that $\langle\!\langle \mathbb{W}_\mu(\Gamma) \rangle\!\rangle = 0$ if the graph Γ is not a simple closed one, that is, if the set $\partial\Gamma \equiv \{i \in (1, ..., N), i$ belongs to an odd number of edges of $\Gamma\}$ is not empty. It follows that

$$\langle\!\langle \tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle = \left[\sum_{\Gamma:\, \partial\Gamma = \varnothing} \langle\!\langle \mathbb{W}_1(\Gamma) \rangle\!\rangle \right]^{\alpha N} \tag{2.7}$$

Coming back to (2.4), we can write

$$\tilde{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = 2^{-N} \sum_{\Gamma_1} \cdots \sum_{\Gamma_p} \mathbb{W}_1(\Gamma_1) \cdots \mathbb{W}_p(\Gamma_p) \sum_{\underline{\sigma}} \sigma_{\Gamma_1} \cdots \sigma_{\Gamma_p} \tag{2.8}$$

and $\sum_{\underline{\sigma}} \sigma_{\Gamma_1} \cdots \sigma_{\Gamma_p} = 0$ if the multigraph $\Gamma_1 \circ \cdots \circ \Gamma_p$ is not a simple closed graph. So the summation over $\underline{\sigma}$ couples the different frequencies μ. We want to extract from $\tilde{\mathbb{Z}}_N$ everything that is not $\underline{\sigma}$ dependent. Then let us call a string any open connected graph where each vertex belongs to no more than two edges: a product of strings will be any graph not containing closed paths. We want to decouple each term in the set $\{\Gamma\}$ of all graphs as a product of strings and simple loops. It is clear that in (2.4),

$$\sigma_\Gamma \equiv \sigma_{\partial\Gamma}, \quad \forall \Gamma \tag{2.9}$$

Then, if Γ_C denotes any product of strings, we can perform in (2.4) the summation over all graphs with fixed "boundary" $\partial\Gamma$, $\partial\Gamma$ denoting any sets of vertices,

$$\sum_{\Gamma} \mathbb{W}_\mu(\Gamma)\sigma_\Gamma = \sum_{\partial\Gamma} \sigma_{\partial\Gamma} \left\{ \sum_{\Gamma:\, \partial\Gamma \text{ fixed}} \mathbb{W}_\mu(\Gamma) \right\} \tag{2.10}$$

We would like to write each term in the last summation of the RHS of (2.10) as a composition of a product of strings Γ_C with a simple closed graph: $\Gamma = \Gamma_C \circ \Gamma'$ with $\partial\Gamma' = \emptyset$. To do that, let us give the following rule: we start from an endpoint of Γ, i.e., a vertex of ∂F belonging to only an edge, and run along the graph until we arrive at a branching point. At that moment we remove the scanned part of the graph: we repeat the operation until $\partial\Gamma$ no longer contains an endpoint, obtaining in such a way a product of strings and a remaining part (only this one if $\partial\Gamma$ does not contain any endpoint at all). If the remaining part is a simple closed graph, then we say that Γ is a "good graph"; otherwise it is a "bad." The remaining part of a bad graph contains at least two vertices which are not simple (i.e., they belong to an odd number of edges: from now on we shall denote such a vertex as n.s.v.). We shall include in the class of bad graph also the graphs where the obtained product of strings contains more than one n.s.v. or one n.s.v. belong to more than three edges. Let us remark that in the simplest closure of a bad graph, i.e., in the closed simple graph obtained from it by adding the smallest possible number of extra vertices and edges, we have

$$\sharp(\text{vertices}) \leqslant \sharp(\text{edges}) + 2 \tag{2.10'}$$

The same could be true also for some good graph, but we ignore this. Now let us consider the difference

$$\sum_{\partial\Gamma} \sigma_{\partial\Gamma} \sum_{\Gamma:\, \partial\Gamma \text{ fixed}} \mathbb{W}_\mu(\Gamma) - \left\{ \sum_{\Gamma:\ \partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \right\} \left[\sum_{\partial\Gamma} \sigma_{\partial\Gamma} \sum^{*}_{\Gamma_C:\, \partial\Gamma_C=\partial\Gamma} \mathbb{W}_\mu(\Gamma_C) \right] \tag{2.11}$$

The starred sum in (2.11) is over graphs Γ_C with at most one n.s.v.; in (2.11) there will appear graphs with a repeated bond [i.e., with some bond (i, j) appearing at least two times] besides all the bad graphs. We shall refer to them globally as to the bad graphs characterized essentially by (2.10′). Graphs in (2.11) could appear with a multiplicity $\nu(\Gamma)$ larger than one. The multiplicity of a graph in (2.11) is equal to the number of ways to decouple it as $\Gamma=\Gamma_C \circ \Gamma'$ with $\partial\Gamma'=\varnothing$, minus one. Let us denote (2.11) by $\mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha)$. Then we write

$$\sum_{\Gamma} \mathbb{W}_\mu(\Gamma)\sigma_\Gamma = \left\{ \sum_{\Gamma:\, \partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \right\} \cdot \left\{ \sum_{\partial\Gamma} \sigma_{\partial\Gamma} \sum_{\Gamma_C:\, \partial\Gamma_C=\partial\Gamma} \mathbb{W}_\mu(\Gamma_C) + \left[\sum_{\Gamma:\, \partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \right]^{-1} \cdot \mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha) \right\} \tag{2.12}$$

with

$$\mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha) = \sum^{**}_{\Gamma} \sigma_{\partial\Gamma} \nu(\Gamma)\, \mathbb{W}\mu(\Gamma) \tag{2.13}$$

and the last summation in (2.13) is over all the graphs described above.

Let us now define

$$\mathbb{Z}_N^{(1)}(\underline{\xi}. \beta, \alpha) = 2^N \prod_\mu \left\{ \left[\prod_{i<j} \cosh \frac{\beta}{N} \mathbb{J}^\mu_{ij} \right] \sum_{\Gamma:\, \partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \right\} \tag{2.14}$$

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = 2^{-N} \sum_{\underline{\sigma}} \prod_\mu \sum_{\partial\Gamma} \sigma_{\partial\Gamma} \sum^{*}_{\Gamma_C:\, \partial\Gamma_C=\partial\Gamma} \mathbb{W}_\mu(\Gamma) \tag{2.15}$$

We will show that $[\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) \cdot \hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)]$ is the main term of $\mathbb{Z}_N(\underline{\xi}, \beta, \alpha)$ in the following sense:

$$\mathbb{Z}_N(\underline{\xi}, \beta, \alpha) = \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)[\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) + \mathbb{R}_N(\underline{\xi}, \beta, \alpha)] \tag{2.16}$$

and $\forall \delta > 0,\ \beta J^2 < 1$

$$\mathbb{P}\text{rob}\left(\left| \frac{\mathbb{R}_N(\underline{\xi}, \beta, \alpha)}{\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha \rangle\rangle} \right| > N^{-3/2+\delta} \right) \leqslant c \cdot N^{-2\delta m}, \qquad \forall m \in \mathbb{Z}_+ \tag{2.17}$$

where

$$\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\rangle = \langle\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\, \hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha \rangle\rangle = \langle\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) \rangle\rangle \qquad (2.17')$$

Equation (2.16) can be also written under the same conditions as

$$\mathbb{Z}_N(\underline{\xi}, \beta, \alpha) = \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)[\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) + \mathbb{R}_N^{(1)}(\underline{\xi}, \beta, \alpha)] \qquad (2.16')$$

where

$$\mathbb{P}\text{rob}(|\mathbb{R}_N^{(1)}(\underline{\xi}, \beta, \alpha)| > N^{-3/2+\delta}) \leqslant c' \cdot \mathbb{N}^{-2\delta m}, \qquad \forall m \in \mathbb{Z}_+ \quad (2.17'')$$

In (2.17) and (2.17′) c and c' are some positive constants depending on β. In the following sections we shall prove otherwise that

$$\frac{\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\, \hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)}{\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\rangle}$$

tends in distribution to the exponential of a Gaussian field. We prove (2.17) and (2.17″) in the Appendix.

Now we are ready to describe our results. We shall compute the mean value of the partition function for large N and prove the sharp definition of the free energy per site in the ergodic region. More exactly, we prove that

$$\lim_{N \to \infty} \left\langle\left\langle \frac{1}{N} \ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \right\rangle\right\rangle = \lim_{N \to \infty} \frac{1}{N} \ln \langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\rangle \qquad (2.18)$$

and the equivalence of (2.18), in the ergodic region, to the vanishing in the limit $N \to \infty$ of the mean value of the order parameter,

$$\tau_N := \frac{2}{N(N-1)} \sum_{i<j} [\langle \sigma_i \sigma_j \rangle_N]^2 \qquad (2.19)$$

where $\langle \cdot \rangle_N$ is the mean with respect the Gibbs distribution for finite N.

Proposition 2.1. Let J^2 be the common variance of the variables ξ_i^μ.

(i) For $\beta, \alpha: J^2\beta < 1$

$$\lim_N \frac{1}{N} \cdot \ln \langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\rangle = \ln 2 - \frac{\alpha}{2} [\ln(1 - \beta J^2) + \beta J^2] \qquad (2.20)$$

From now on let $\mathbb{Q}(\beta)$ denote such a limit.

(ii) For all β, α: $\sqrt{\alpha}\,\beta J^2/(1-\beta J^2)<1$,

$$\lim_N \langle\!\langle \tau_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle = 0 \tag{2.21}$$

and (iii)

$$\lim_N \frac{1}{N} \cdot \langle\!\langle \ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle = \lim_N \frac{1}{N} \cdot \ln \langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle \tag{2.22}$$

and (iv) The free energy per site converges to its mean value in probability,

$$\lim_N \mathbb{P}\mathrm{rob}\left[\left|\frac{1}{N} \cdot \ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) - \ln 2 + \frac{\alpha}{2}\,[\ln(1-\beta J^2) + \beta J^2]\right| > \delta\right] = 0$$

$$\forall \delta > 0 \quad \text{and also in } L^p \text{ sense} \quad \forall p < \infty \tag{2.23}$$

Let us remark that the limit (2.20) coincides, for small β, with that found by Amit *et al.*[2] using the replica symmetry technique.

Moreover, we shall study the fluctuations of the free energy, namely

$$\ln\left[\frac{\mathbb{Z}_N(\underline{\xi}, \beta, \alpha)}{\langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle}\right]$$

These fluctuations are of order 1 and converge in distribution to a field which is the sum of two orthogonal shifted Gaussian variables.

Proposition 2.2.

(i) For all α, β: $\sqrt{\alpha}\,\beta J^2/(1-\beta J^2)<1$, $\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)$ tends in distribution to the log-normal variable

$$\hat{\mathbb{Q}} = \exp(v - \tfrac{1}{2}\langle v^2 \rangle) \tag{2.24}$$

where v is a Gaussian variable with covariance

$$\langle v^2 \rangle = -\frac{1}{2}\left\{\ln\left[1 - \alpha\left(\frac{\beta J^2}{1-\beta J^2}\right)^2\right] + \alpha\left(\frac{\beta J^2}{1-\beta J^2}\right)^2\right\} \tag{2.25}$$

(ii) For α, β: $\beta J^2 < 1$,

$$\frac{\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)}{\langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle}$$

tends in distribution to the variable $\mathbb{Q} = \exp(u - \frac{1}{2}\langle u^2 \rangle)$, where u is a Gaussian variable with zero mean and variance

$$\langle u^2 \rangle = \frac{\alpha\beta^4 J^4}{8[1-(\beta J^2)]^2} (\langle\langle (\xi_1^1)^4 \rangle\rangle - J^4) \tag{2.26}$$

So the fluctuations due to the field v seem to "diverge" before the ones due to the field u. Let us call u and v the "warm" and "cold" fields, respectively.

3. THE MEAN VALUE OF THE PARTITION FUNCTION AND THE LIMITING COLD FIELD

Let us now consider the variable $\prod_\mu \sum_{\Gamma:\partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma)$. As was pointed out in ref. 1, each graph in the sum is a product of simple loops but not any product of loops is contained in it. Nevertheless let us consider as in ref. 1 the variable

$$\mathbb{W}_N(\underline{\xi}, \beta, \alpha) = \sum_{\mu=1}^{\alpha N} \sum_{\gamma:|\gamma|\geqslant 3} \mathbb{W}_\mu(\gamma) \tag{3.1}$$

where the sum runs over all the simple loops.

Before going on, it is convenient to consider the term inside the brackets in (2.2),

$$\prod_\mu \prod_{i<j} \cosh\left(\frac{\beta}{N} \mathbb{J}_{ij}^\mu\right) = \exp\left\{\sum_\mu \sum_{i<j} \left[\frac{\beta^2}{2N^2} (\mathbb{J}_{ij}^\mu)^2 - \frac{2}{4!} \frac{\beta^4}{N^4} (\mathbb{J}_{ij}^\mu)^4\right] + o\left(\frac{1}{N}\right)\right\} \tag{3.2}$$

Let us write $\sum_{i<j} (\mathbb{J}_{ij}^\mu)^2 \beta^2/2N^2$ in the form

$$\sum_{i<j} \frac{\beta^2}{2N^2} \mathbb{J}_{ij}^\mu)^2 := \sum_{\gamma:|\gamma|=2} \mathbb{W}_\mu(\gamma) + o\left(\frac{1}{N^4}\right) \tag{3.3}$$

A lemma similar to Lemma 3.1 in ref. 1 can be now given with respect to the variable

$$\tilde{\mathbb{W}}_N(\underline{\xi}, \beta, \alpha) = \sum_\mu \sum_{\gamma:|\gamma|\geqslant 2} [\mathbb{W}_\mu(\gamma) - \langle\langle \mathbb{W}_\mu(\gamma) \rangle\rangle] := \sum_\mu \sum_{\gamma:|\gamma|\geqslant 2} \tilde{\mathbb{W}}_\mu(\gamma) \tag{3.4}$$

Lemma 3.1. For $\beta J^2 < 1$, we have the following results.

(i) $\tilde{\mathbb{W}}_N(\underline{\xi}, \beta, \alpha)$ converges in distribution to a Gaussian variable u with covariance given by (2.26).

(ii) $\sum_{\mu}[\sum_{\gamma:|\gamma|\geqslant 2}\tilde{\mathbb{W}}_{\mu}(\gamma)]^2$ converges in distribution and in L^2 sense to the constant $\langle u^2\rangle$.

(iii) The following holds:

$$\lim_{N}\operatorname{Prob}(\max_{\mu,\gamma}|\tilde{\mathbb{W}}_{\mu}(\gamma)|>N^{-2+\varepsilon})=0 \qquad \forall\varepsilon>0 \tag{3.5}$$

(iv) The following holds:

$$\lim_{N}\operatorname{Prob}\left(\max_{\mu}|\sum_{\gamma}\tilde{\mathbb{W}}_{\mu}(\gamma)|>N^{-1/2+\varepsilon}\right)=0 \qquad \forall\varepsilon>0 \tag{3.6}$$

(v) Then by implication,

$$\sum_{\mu}\left|\sum_{\gamma}\tilde{\mathbb{W}}_{\mu}(\gamma)\right|^k\to 0 \quad \text{in distribution} \quad \text{as } N\to\infty \quad \text{and } \forall k>2 \tag{3.7}$$

Proof. The proof moves along the same lines as in ref. 1, but the mechanism of the construction of the Gaussian field is different. Let us consider

$$\left\langle\!\!\left\langle\left[\sum_{\mu}\sum_{\gamma}\tilde{\mathbb{W}}_{\mu}(\gamma)\right]^2\right\rangle\!\!\right\rangle=\sum_{\mu_1\mu_2}\sum_{\gamma_1\gamma_2}\langle\!\langle\tilde{\mathbb{W}}_{\mu_1}(\gamma_1)\,\tilde{\mathbb{W}}_{\mu_1}(\gamma_2)\rangle\!\rangle \tag{3.8}$$

If $\mu_1\neq\mu_2$ and/or $\gamma_1\cap\gamma_2=\varnothing$, the RHS of (3.8) = 0. If $\gamma_1\cap\gamma_2$ contains more than one point (vertex), it is not hard to see that the contribution will be zero as N goes to infinity (the contribution goes like c/N). The only surviving contribution comes from terms with $\mu_1=\mu_2$ and $|\gamma_1\cap\gamma_2|=1$. Moreover,

$$\sum_{\mu}\sum_{\gamma_1\cdots\gamma_n}\langle\!\langle\tilde{\mathbb{W}}_{\mu}(\gamma_1)\cdots\tilde{\mathbb{W}}_{\mu}(\gamma_n)\rangle\!\rangle=\alpha N\sum_{\gamma_1\cdots\gamma_n}\langle\!\langle\tilde{\mathbb{W}}_{1}(\gamma_1)\cdots\tilde{\mathbb{W}}_{1}(\gamma_n)\rangle\!\rangle \tag{3.9}$$

The number of graphs grows like $N^{\sharp(\text{vertices})}$, while the contribution of each graph goes like $N^{-|\Gamma|}$, $\Gamma=\gamma_1\circ\cdots\circ\gamma_n$.

It is easy to see that (3.9) goes to zero if $n>2$ [let us remember that $\sum_{\gamma_1\gamma_2}\langle\!\langle\tilde{\mathbb{W}}_{\mu}(\gamma_1)\,\tilde{\mathbb{W}}_{\mu}(\gamma_2)\rangle\!\rangle\approx c/N$]. So we conclude that

$$\left\langle\!\!\left\langle\left[\sum_{\mu}\sum_{\gamma}\tilde{\mathbb{W}}_{\mu}(\gamma)\right]^n\right\rangle\!\!\right\rangle\to\sharp[\text{pairings}(1\cdots n)]\left[\sum_{\mu}\sum_{\substack{\gamma,\gamma'\\|\gamma\cap\gamma'|=1}}\langle\!\langle\tilde{\mathbb{W}}_{\mu}(\gamma)\,\tilde{\mathbb{W}}_{\mu}(\gamma')\rangle\!\rangle\right]^{n/2} \tag{3.10}$$

The remaining points of the lemma are proved along the same lines as in ref. 1, and we omit their proof. We perform direct computations in the next section where we study the warm field coming from $\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)$. Observe that

$$\sum_{\mu} \sum_{\gamma:|\gamma| \geqslant 2} \langle\!\langle \mathbb{W}_\mu(\gamma) \rangle\!\rangle = \alpha N \sum_{k=2}^{N} \frac{N^k - \frac{1}{2}k(k-1)N^{k-1}}{2k} \left(\frac{\beta J^2}{N}\right)^k + o\left(\frac{1}{N}\right) \tag{3.11}$$

Then, for large N

$$\text{LHS (3.11)} = -\frac{\alpha N}{2}[\ln(1-\beta J^2) + \beta J^2] - \frac{\alpha}{4}\frac{\beta J^2}{(1-\beta J^2)^2} + o\left(\frac{1}{N}\right) \tag{3.12}$$

and

$$\begin{aligned}
&\left\langle\!\!\left\langle \left[\sum_{\mu}\sum_{\gamma} \tilde{\mathbb{W}}_\mu(\gamma)\right]^2 \right\rangle\!\!\right\rangle \\
&\quad = \alpha N \sum_{\gamma_1\gamma_2:\, |\gamma_1 \cap \gamma_2| = 1} [\langle\!\langle \mathbb{W}_1(\gamma_1)\, \mathbb{W}_1(\gamma_2) \rangle\!\rangle \\
&\qquad - \langle\!\langle \mathbb{W}_1(\gamma_1) \rangle\!\rangle \langle\!\langle \mathbb{W}_1(\gamma_2) \rangle\!\rangle] + o\left(\frac{1}{N}\right) \\
&\quad = \frac{\alpha}{2} N \sum_{k_1=2}^{N-1} \sum_{k_2=1}^{N-k_1} \binom{N}{k_1} \frac{k_1!}{2k_1} \\
&\qquad \binom{N-k_1}{k_2} \frac{k_2!}{2k_2} k_1 k_2 \left(\frac{\beta}{N}\right)^2 \left(\frac{\beta J^2}{N}\right)^{k_1+k_2-1} (\langle\!\langle \xi_1^4 \rangle\!\rangle - \langle\!\langle \xi_1^2 \rangle\!\rangle^2) + o\left(\frac{1}{N}\right) \\
&\quad = \frac{\alpha \beta^4 J^4}{8(1-\beta J^2)^2} (\langle\!\langle \xi_1^4 \rangle\!\rangle - J^4) + o\left(\frac{1}{N}\right)
\end{aligned} \tag{3.13}$$

In (3.13) the factor $k_1 \cdot k_2$ in the numerator takes into account the number of ways γ_1 *and* γ_2 have to intersect each other, while the factor 1/2 in front is because each loop has to appear only one time. As we shall see,

$$-\alpha \frac{N}{2}[\ln(1-\beta J^2) + \beta J^2] - \frac{\alpha}{4}\frac{\beta J^2}{(1-\beta J^2)^2} + \frac{1}{2}\frac{\alpha \beta^4 J^4}{8(1-\beta J^2)^2} \langle\!\langle \xi_1^4 \rangle\!\rangle - J^4)$$

is nothing else than $\ln \langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle - N \ln 2$ for large N. To see that, let us consider

$$\begin{aligned}
\langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle &= 2^N \left[\left\langle\!\!\left\langle \prod_{i<j} \left(\cosh \frac{\beta}{N} J_{ij}^1\right) \sum_{\Gamma:\partial\Gamma = \varnothing} \mathbb{W}_1(\Gamma) \right\rangle\!\!\right\rangle \right]^{\alpha N} \\
&= \langle\!\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) \rangle\!\rangle
\end{aligned} \tag{3.14}$$

By (3.3) we can also write

$$\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) = \prod_{\mu} \sum_{\Gamma : \partial\Gamma = \varnothing}^{*} \mathbb{W}_\mu(\Gamma) \tag{3.15}$$

where the summation is over all closed, not necesserily simple graphs: some of them could contain edges with a multiplicity larger than one, but less than or equal to three (but of course vertices are all simple). It follows that

$$\langle\!\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) \rangle\!\rangle = 2^N \left[\prod_{\mu} \sum_{m} \sum_{\gamma_1 \cdots \gamma_m}^{*} \langle\!\langle \mathbb{W}_\mu(\gamma_1) \cdots \mathbb{W}_\mu(\gamma_m) \rangle\!\rangle \right] \tag{3.16}$$

where the summation is over m simple loops $\gamma_1 \neq \cdots \neq \gamma_m$, under the condition that not more than three of them can have common edges. By removing any constraint in the sum over $\gamma_1 \cdots \gamma_m$, it is clear from Lemma 3.1 (see also Lemma 3.2 below) that we add contributions which are vanishingly small to the mean value of $\mathbb{Z}_N^{(1)}$ (at least vanishing like N^{-4}). Then,

$$\langle\!\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha) \rangle\!\rangle = 2^N \prod_{\mu} \left\{ \sum_{m} \frac{1}{m!} \left\langle\!\!\left\langle \left[\sum_{\gamma} (\widetilde{\mathbb{W}}_\mu(\gamma) + \langle\!\langle \mathbb{W}_\mu(\gamma) \rangle\!\rangle) \right]^m \right\rangle\!\!\right\rangle \right.$$
$$\left. \times \left[1 + o\left(\frac{1}{N^4} \right) \right] \right\} \tag{3.17}$$

By Lemma 3.1,

$$\text{LHS (3.17)} = 2^N \prod_{\mu} \left\{ \sum_{m} \sum_{k=0}^{[m/2]} \frac{1}{(m-2k)!\, k!} \right.$$
$$\times \left[\frac{1}{2} \sum_{\substack{\gamma_1 \gamma_2 \\ |\gamma_1 \cap \gamma_2| = 1}} \langle\!\langle \widetilde{\mathbb{W}}_\mu(\gamma_1)\, \widetilde{\mathbb{W}}_\mu(\gamma_2) \rangle\!\rangle \right]^k$$
$$\left. \times \left[\sum_{\gamma} \langle\!\langle \mathbb{W}_\mu(\gamma) \rangle\!\rangle \right]^{m-2k} \left[1 + o\left(\frac{1}{N^4} \right) \right] \right\} \tag{3.18}$$

So for very large N we can write

$$\langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle$$
$$= 2^N \exp\left\{ -\frac{\alpha N}{2} [\ln(1 - \beta J^2) + \beta J^2] - \frac{\alpha}{4} \frac{\beta J^2}{(1 - \beta J^2)^2} \right.$$
$$\left. \times \frac{1}{2} \frac{\alpha \beta^4 J^4}{8(1 - \beta J^2)^2} (\langle\!\langle \xi_1^4 \rangle\!\rangle - J^4) + o\left(\frac{1}{N^3} \right) \right\} \tag{3.19}$$

Remark. It is well known that, for fixed p and large T,

$$\lim_N \ln[2^{-N}\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha=0)\rangle\rangle] = 0$$

Let us now observe that

$$\begin{aligned}&\sum_\mu \prod_\gamma [1+\mathbb{W}_\mu(\gamma)] - 2^{-N}\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\\ &= \sum_{h=1}^{\alpha N} \left[\sum_{\Gamma:\partial\Gamma=\varnothing}^{**} \mathbb{W}_h(\Gamma)\right] \prod_{\mu=1}^{h-1} \sum_{\Gamma:\partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \prod_{\mu=h+1}^{\alpha N} \prod_\gamma [1+\mathbb{W}_\mu(\gamma)]\end{aligned} \tag{3.20}$$

where the second summation on the RHS of (3.20) is over all closed graphs, each of them containing at least one edge with a multiplicity larger than or equal to four.

Lemma 3.2. For $\beta J^2 < 1$,

$$[\langle\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\rangle\rangle]^{-1} \left\{\prod_\mu \prod_\gamma [1+\mathbb{W}_\mu(\gamma)] - 2^{-N}\mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\right\} \to 0$$

in probability when N goes to infinity.

Proof. The lemma follows easily from

$$\begin{aligned}&\operatorname*{Prob}_{\underline{\xi}}\left\{\left|\sum_{\Gamma:\partial\Gamma\neq\varnothing}^{**} \mathbb{W}_1(\Gamma)\right| > N^{-5}\right\}\\ &\leqslant N^5 \left[\sum_{m=4}^{[N/2]} \frac{1}{m!} \sum_{\gamma_1\cdots\gamma_m}^{**} \langle\langle \mathbb{W}_1(\gamma_1)\cdots\mathbb{W}_1(\gamma_m)\rangle\rangle\right] \leqslant \frac{C}{N}\end{aligned} \tag{3.21}$$

with some positive constant C, and

$$\operatorname*{Prob}_{\underline{\quad}}\left\{\frac{\prod_\mu \prod_\gamma [1+\mathbb{W}_\mu(\gamma)]}{2^{-N}\langle\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\rangle\rangle} > N\right\} \leqslant \frac{C_1}{N} \tag{3.21$'$}$$

for some positive constant C_1.

Lemma 3.3. For $\beta J^2 < 1$,

$$[2^{-N}\langle\langle \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\rangle\rangle]^{-1} \prod_\mu \prod_\gamma [1+\mathbb{W}_\mu(\gamma)] \to \exp\left(u - \frac{1}{2}\langle u^2\rangle\right)$$

in distribution, where u is the Gaussian variable of Proposition 2.2.

Proof. From Lemma 3.1 it is easy to prove that

$$\forall k>1, \qquad \sum_{\mu}\sum_{\gamma}|\mathbb{W}_\mu(\gamma)|^k \to 0 \quad \text{in distribution}$$

so that

$$\begin{aligned}
&[2^{-N}\langle\!\langle \mathbb{Z}_N^{(1)}(\underline{\xi},\beta,\alpha)\rangle\!\rangle]^{-1}\prod_{\mu}\prod_{\gamma}[1+\mathbb{W}_\mu(\gamma)]\\
&\quad=[2^{-N}\langle\!\langle \mathbb{Z}_N^{(1)}(\underline{\xi},\beta,\alpha)\rangle\!\rangle]^{-1}\exp\left[\sum_{\mu}\sum_{\gamma}\mathbb{W}_\mu(\gamma)+o\left(\frac{1}{N}\right)\right]\\
&\quad=\exp\left[\sum_{\mu}\sum_{\gamma}\tilde{\mathbb{W}}_\mu(\gamma)-\frac{1}{2}\sum_{\mu}\sum_{\gamma_1\gamma_2:|\gamma_1\cap\gamma_2|=1}\langle\!\langle\tilde{\mathbb{W}}_\mu(\gamma_1)\,\tilde{\mathbb{W}}_\mu(\gamma_2)\rangle\!\rangle\right] \qquad (3.22)
\end{aligned}$$

So we have proved point (i) of Proposition 2.1 and point (ii) of Proposition 2.2.

4. FLUCTUATION OF THE TOTAL FREE ENERGY IN THE ERGODIC REGION

Let us now consider the variable

$$\hat{\mathbb{Z}}_N(\underline{\xi},\beta,\alpha)=2^{-N}\sum_{\underline{\sigma}}\prod_{\mu}\sum_{\Gamma_C}^{*}\mathbb{W}_\mu(\Gamma)\sigma_{\Gamma_C} \qquad (4.1)$$

The second summation in (4.1) is over all products of strings (i.e., graphs not including closed paths) containing at most one nonsimple vertex and this one with not more than three edges coming out from it. Let us start by considering terms containing only one string with (fixed) endpoints i, j. The sum over all these strings looks like

$$\begin{aligned}
\mathbb{A}_{i,j}^{\mu}&=\sum_{h=0}^{N-2}\sum_{i_1\neq i_2\neq\cdots\neq i_h\neq(i,j)}\tanh\left(\frac{\beta}{N}J_{ii_1}^{\mu}\right)\left[\prod_{l=1}^{h-1}\tanh\left(\frac{\beta}{N}J_{i_li_{l+1}}^{\mu}\right)\right]\\
&\quad\times\tanh\left(\frac{\beta}{N}J_{i_hj}^{\mu}\right)\\
&=\sum_{h=0}^{N-2}\left(\frac{\beta}{N}\xi_i^\mu\xi_j^\mu\right)\prod_{l=1,\dots,h}^{**}\sum_{i_l}^{**}\frac{\beta}{N}(\xi_{i_l}^\mu)^2(1+r_{1,N}) \qquad (4.2)
\end{aligned}$$

where

$$\mathbb{P}\text{rob}(|r_{1,N}|>N^{-2+\delta})\leqslant N^{-k\delta} \qquad \forall k>0 \qquad (4.2')$$

and the double star in (4.2) means $i_1\neq i_2\neq\cdots\neq i_h\neq(i,j)$.

The sum $\sum_{i_1 \neq i_2 \neq \cdots \neq i_h}$ can be written as a combination of "free" sums, so we get

$$\prod_{l=1}^{b} \sum_{i_l}^{**} \left[\frac{\beta}{N}(\xi_{ij}^{\mu})^2\right](1+r_{1,N}) = \prod_{l=1}^{h} \sum_{i_l} \left[\frac{\beta}{N}(\xi_{i_l}^{\mu})^2\right](1+r_{1,N})(1+r_{2,N}) \tag{4.3}$$

where

$$\mathbb{P}\text{rob}(|r_{2,N}| > N^{-1+\delta_1}) \leqslant N^{-1+\delta_1}) \leqslant N^{-k\delta_1} \quad \forall k>0 \text{ and some } \delta_1 > 0$$

Finally, by the law of large numbers we write

$$\begin{aligned} &\frac{\beta}{N}\sum_{j=1}^{N} (\xi_i^{\mu})^2 = \beta J^2 + r_{3,N} \\ &\quad \text{with} \quad \mathbb{P}\text{rob}(|r_{3,N}| > N^{-1/2+\delta_2}) \leqslant N^{-m\delta_2} \\ &\quad \forall m > 0 \quad \text{and for some } \delta_2 > 0 \quad \text{when } \beta J^2 < 1 \end{aligned} \tag{4.4}$$

So it is not hard to see that for $\beta J^2 < 1$

$$\mathbb{A}_{ij}^{\mu} = \frac{\mathbb{W}_{\mu}(J_{ij}^{\mu})}{1-\beta J^2}\left(1 + \frac{\hat{r}_N}{1-\beta J^2}\right) \tag{4.5}$$

with

$$\mathbb{P}\text{rob}(|\hat{r}_N| > N^{-1/2+\delta_3}) \leqslant N^{-k\delta_3} \quad \forall k > 0 \text{ and for some } \delta_3 > 0 \tag{4.5$'$}$$

It could seem dangerous to perform the product over μ with such a bad estimate; nevertheless $\hat{r}_N$ appears in the "renormalization factor" and it will not play any role at all. Of course if ξ_i^{μ} take values ± 1, this difficulty does not appear. In considering terms which are products of strings, we meet three possibilities: (1) There exists (only) one nonsimple common vertex, (2) there exist common simple vertices, (3) all the strings are not connected. Looking at point 2, each term contains somewhere a factor

$$\left(\frac{\beta}{N}\right)^2 \sum_i (\xi_i^{\mu})^4$$

which goes like $c_1 N^{-1+\delta_4}$ with a probability larger than $1 - c_2 N^{-k\delta_4}$, $\forall k > 0$, and some positive constants δ_4, c_1, c_2. We shall include terms of point 2 in point 3 as a vanishingly small correction. About point 1, we glance, for instance, at terms which are a product of two strings with endpoints (i, j, k). Their sum will give the contribution

$$\sum_{h_1=0}^{N-3} \sum_{h_2=0}^{N-3-h_1} \left(\frac{\beta}{N}\right)^{3/2} \xi_i^\mu \xi_j^\mu \xi_k^\mu h_1 \sum_{i_1=j_1\neq i_2\neq \cdots \neq i_{h_i}\neq j_2\neq \cdots \neq j_{h_2}} \left[\left(\frac{\beta}{N}\right)^{3/2} (\xi_{i_1}^\mu)^3\right] \times\left[\left(\frac{\beta}{N}\right)(\xi_{i_2}^\mu)^2\right]\cdots\left[\left(\frac{\beta}{N}\right)(\xi_{j_{h_2}}^\mu)^2\right] \tag{4.6}$$

By the law of large numbers we can write such a contribution as

$$\left(\frac{\beta}{N}\right)^{3/2} \xi_i^\mu \xi_j^\mu \xi_k^\mu \cdot q_N$$

where q_N is a stochastic variable such that

$$\mathbb{P}\text{rob}(|q_N| > N^{-1+\delta_5}) \leqslant N^{-k\delta_5} \qquad \forall k>0 \text{ and some } \delta_5>0$$

These graphs will not give contributions (as N goes to infinity) because, as will be clear below, to get a closed graph with simple vertices, at least two of them have to appear inside, so that the whole contribution will go as $CN^{-2(1-\delta)}$ with some constant C. So we shall consider only the terms of possibility 3. Looking at these terms, let us observe that each product of strings can be seen as a combination of products of "free" strings (i.e., without any constraint among the indices of vertices belonging to different strings). It is not hard to realize that, for instance, the sum of all products of two strings with endpoints (i, j)-(h, k) can be written as

$$\mathbb{A}_{ij}^\mu \mathbb{A}_{hk}^\mu (1+\bar{r}_N) \qquad \text{where} \quad \mathbb{P}\text{rob}(|\bar{r}_N| > N^{-1+\delta_6}) \leqslant N^{-m\delta_6}$$
$$\forall m \in \mathbb{Z}_+ \quad \text{and some } \delta_6>0 \tag{4.7}$$

Moreover, all graphs constructed by these terms give a contribution going like N^{-1} (see Remark below). Then, let us denote by $\tilde{\Gamma}$ a graph which is a product of disjoined edges:

$$\mathbb{A}^\mu(\tilde{\Gamma}) = \prod_{(i,j)\in\tilde{\Gamma}} \mathbb{A}_{ij}^\mu$$

so we can look at $\hat{\mathbb{Z}}_N$ as

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = 2^{-N} \sum_{\underline{\sigma}} \prod_\mu \sum_{\tilde{\Gamma}} \mathbb{A}^\mu(\tilde{\Gamma})\sigma_{\tilde{\Gamma}}$$
$$= 2^{-N} \sum_{\tilde{\Gamma}_1\cdots\tilde{\Gamma}_p} \mathbb{A}^1(\tilde{\Gamma}^1)\cdots\mathbb{A}^p(\tilde{\Gamma}^p)\sum_{\underline{\sigma}} \sigma_{\tilde{\Gamma}^1}\cdots\sigma_{\tilde{\Gamma}^p} \tag{4.8}$$

But

$$\sum_{\sigma} \sigma_{\tilde{\Gamma}^1} \cdots \sigma_{\tilde{\Gamma}^p} = 0$$

if the multigraph $\tilde{\Gamma}_1 \circ \cdots \circ \tilde{\Gamma}_p$ is not closed with all simple vertices. So we get

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = \sum_{\tilde{\Gamma}_1 \cdots \tilde{\Gamma}_p : \partial(\tilde{\Gamma}_1 \circ \cdots \circ \tilde{\Gamma}_p) = \varnothing} \mathbb{A}^1(\tilde{\Gamma}^1) \cdots \mathbb{A}^p(\tilde{\Gamma}^p) \tag{4.8'}$$

Let us observe that the multigraph $\tilde{\Gamma}_1 \circ \cdots \circ \tilde{\Gamma}_p$ has only simple vertices but could contain repeated edges: in such a case the variables associated to the same edges have to belong to different values of the index μ. Let us define a *nude graph* as any closed graph where each vertex belongs to an even number of links. Let us consider the $\frac{1}{2}N(N-1)p$ variables $\mathbb{A}^{\mu}_{ij}$. A *marked graph* is a function

$$M: \quad \{\Gamma\} \to \{M(\Gamma)\}$$

which assigns to each edge (i, j) of a nude graph Γ a variable $\mathbb{A}^{\mu}_{ij}$ in such a way that two edges with a common vertex cannot have variables with the same index μ.

Then,

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = \sum_M \sum_{\Gamma : \partial\Gamma = \varnothing} M(\Gamma) \tag{4.9}$$

where $\sum_\Gamma$ is the summation over all the nude graphs and $\sum_M$ is the summation over all the possible ways to dress a nude graph.

Remark. The number of ways to dress a nude graph goes as $\lesssim N^{\sharp(\text{edges})}$: but if two edges have the same frequence μ, then this number decreases by a factor $1/N$. Let γ be a simple nude loop. As the function M is well defined on such a graph, we can consider the variable (the star is to indicate that the summation is over the simple loops)

$$\mathbb{V}_N = \sum_{M, \gamma}^{*} M(\gamma) \tag{4.10}$$

Let us remark at once that $\langle\!\langle \mathbb{V}_N \rangle\!\rangle = 0$ and $\forall \gamma, \gamma'$ such that $M(\gamma) \neq M(\gamma')$, the only case with $\langle\!\langle M(\gamma)\, M(\gamma') \rangle\!\rangle \neq 0$ is when $\gamma = \gamma'$ and $M(\gamma) \circ M(\gamma')$ is a closed chain each ring of which contains the same kinds of edges (w.r.t. the index μ):

$$M(\gamma) = \mathbb{A}^{\mu_1}_{i_1, i_2} \mathbb{A}^{\mu_2}_{i_2, i_3} \mathbb{A}^{\mu_1}_{i_3, i_4} \cdots \mathbb{A}^{\mu_1}_{i_n, i_1}; \qquad M(\gamma') = \mathbb{A}^{\mu_2}_{i_1, i_2} \mathbb{A}^{\mu_1}_{i_2, i_3} \cdots \mathbb{A}^{\mu_2}_{i_n, i_1}$$

and n has to be larger than or equal to four.

But these graphs do not give any contribution as $N \to \infty$, as we shall see.

So we are in a situation very similar to the one in ref. 1. Here each edge contains a factor N^{-1} *instead* of $1/\sqrt{N}$, but we have a larger multiplicity of the graphs.

Lemma 4.1. For $\sqrt{\alpha}\,\beta J^2/(1-\beta J^2)<1$ we have the following results.

(i) $\mathbb{V}_N$ tends in distribution to the gaussian variable v, with covariance given by (2.25).

(ii) $\sum^*_{M,\gamma} [M(\gamma)]^2 \to \langle v^2 \rangle$ as $N \to \infty$, both in distribution and in L^2 sense.

(iii) We have

$$\mathbb{P}\text{rob}[\max_{M,\gamma} |M(\gamma)| > N^{-2+\varepsilon}] \to 0 \quad \text{as } N \to \infty, \quad \forall \varepsilon > 0 \tag{4.11}$$

Then by implication

$$\sum_{M,\gamma} |M(\gamma)|^k \to 0 \qquad \text{in distribution} \quad \forall k > 2 \tag{4.12}$$

Proof. We have

$$\begin{aligned}\langle\!\langle \mathbb{V}_N^2 \rangle\!\rangle = {} & \sum_{k=2}^{N} \sum_{M_k, \gamma_k} \langle\!\langle [M_k(\gamma_k)]^2 \rangle\!\rangle \\ & + \sum_{k=4}^{N} \sum^{*}_{M_k, M'_k, \gamma_k} \langle\!\langle M_k(\gamma_k)\, M'_k(\gamma_k) \rangle\!\rangle \end{aligned} \tag{4.13}$$

where γ_k is a graph with $|\gamma_k| = k$, M_k is the restriction of the function M on $\{\gamma_k\}$, and the second summation in (4.13) is over the marked graphs described above.

Now,

$$\begin{aligned}\sum_{\gamma_k} \langle\!\langle [M_k(\gamma_k)]^2 \rangle\!\rangle \simeq {} & \frac{1}{2k} \left[\frac{\beta}{N(1-\beta J^2)} \right]^{2k} \\ & \times \sum_{i_1 \neq \cdots \neq i_k} \langle\!\langle [\xi_{i_2}^{\mu_k} \xi_{i_1}^{\mu_1}]^2 [\xi_{i_2}^{\mu_1} \xi_{i_2}^{\mu_2}]^2 \cdots [\xi_{i_k}^{\mu_k} \xi_{i_k}^{\mu_{k-1}}]^2 \rangle\!\rangle \end{aligned} \tag{4.14}$$

where $\mu_1 \cdots \mu_k$ depend on M_k, but $\mu_i \neq \mu_{i+1}$, $\forall i = 1,..., k$ [$\simeq$ means we consider only the first term in the development of $\mathrm{tgh}(x)$]. Then,

$$\sum_{\gamma_k} \langle\!\langle [M_k(\gamma_k)]^2 \rangle\!\rangle \simeq \frac{1}{2k} \left[\frac{\beta J^2}{N(1-\beta J^2)} \right]^{2k} \binom{N}{k} k! \tag{4.15}$$

Now we have to sum over all the ways to dress a nude loop γ_k. It is easy to see that the number of these ways is

$$(\alpha N)^k \left(1 - \frac{1}{\alpha N}\right)^{k-2} \left(1 - \frac{1}{\alpha N}\right)^2 \tag{4.16}$$

It is clear otherwise that for the second term in (4.13) the sum over M_k goes as $(\alpha N)^2$. Finally we get, for very large N,

$$\langle\!\langle \mathbb{V}_N^2 \rangle\!\rangle = -\frac{1}{2} \left\{ \ln \left[1 - \frac{\alpha(\beta J^2)^2}{(1-\beta J^2)^2} \right] + \frac{\alpha(\beta J^2)^2}{(1-\beta J^2)^2} \right\} + o\left(\frac{1}{N}\right) \tag{4.17}$$

Following ref. 1, we put $\mathbb{V}_N = \mathbb{V}_N^{\leqslant k} + \mathbb{V}_N^{>k}$, where $\mathbb{V}_N^{>k}$ is constructed with loops γ: $|\gamma| > k$. From (4.15) and (4.16) it is clear that

$$\|\mathbb{V}_N^{>k}\|_2 \leqslant \varepsilon_k \qquad \text{uniformly in } N \text{ with } \varepsilon_k \to 0 \text{ as } k \to \infty$$
$$\text{and} \quad \lim_{k\to\infty} \lim_{N\to\infty} \langle\!\langle [\mathbb{V}_N^{\leqslant k}]^2 \rangle\!\rangle = \langle v^2 \rangle$$

(see criterion C3 in ref. 1). Now we want to show that the quantities

$$\mathbb{R}_{k,n}(N) = \langle\!\langle [\mathbb{V}_N^{\leqslant k}]^n \rangle\!\rangle - \sharp[\text{pairings of } \{1 \cdots n\}] \langle\!\langle [\mathbb{V}_N^{\leqslant k}]^2 \rangle\!\rangle^{n/2} \tag{4.18}$$

vanishes as $N \to \infty$ $\forall k, n$. The proof is essentially the same as in ref. 1. We have

$$\mathbb{R}_{k,n}(N) \sum_{(M,\gamma_1),...(M,\gamma_n): |\gamma_i| \leqslant k} \langle\!\langle M(\gamma_1) \cdots M(\gamma_n) \rangle\!\rangle$$
$$- \sum_{\substack{\text{pairings} \\ \text{of } \{1 \cdots n\}}} \sum_{\substack{(M,\gamma_1),...(M,\gamma_n) \\ |\gamma_i| \leqslant k}} \langle\!\langle M(\gamma_{i_1}) M(\gamma_{j_1}) \rangle\!\rangle \cdots \langle\!\langle M(\gamma) M(\gamma) \rangle\!\rangle \tag{4.19}$$

Let us consider the multigraph $\tilde{M}(\Gamma) := M(\gamma_1) \circ \cdots \circ M(\gamma_n)$. Here the tilde means that the multigraph Γ could be not "well dressed" [$M(\gamma_1 \circ \gamma_2) \neq M(\gamma_1) \circ M(\gamma_2)$]. The only possibility for a bond J_{ij}^μ to appear an odd number of times is inside a clased "sausage" where the number of bonds and the set of indices μ are the same in each segment. When the set of dressed loops $\{M(\gamma_i)\}$ in the first term of (4.19) forms $n/2$ nonintersecting couples of equal loops [here by nonintersecting we mean that if two "double loops" have a common vertex, then the four pairs of edges coming out from the vertex have different values of μ, and two loops could be "equal" also if they appear in the last summation in the second term of (4.13)], then this term is also contained in the second term of (4.19) up to a permutation. The remaining contributions to $\mathbb{R}_{k,n}(N)$ are given by multigraphs with at least a vertex with eight links (with only two different values of μ) or two vertices with six links. Graphs with an odd number of bonds are included in these sets. As in ref. 1, let us write

$$|\mathbb{R}_{k,n}(N)| \leqslant \sum_{\tilde{M},\Gamma}^{*} C(\tilde{M}(\Gamma)\, \omega(\tilde{M}(\Gamma)) \tag{4.20}$$

where the starred summation is over all multigraphs described above, and

$$\omega(\tilde{M}(\Gamma)) \leqslant \left(\frac{\beta \langle\!\langle |\xi_1^1|^n \rangle\!\rangle^{1/n}}{1-\beta J^2} \right)^{|\Gamma|} N^{-|\Gamma|} \tag{4.21}$$

For each n, k there is only a uniformly finite number of equivalence classes of graphs $\tilde{M}(\Gamma)$ modulo permutations of $\{1,..., n\}$. If $n(\Gamma)$ is the number of vertices, the weight of each graph of each class decays like $N^{-|\Gamma|}$, while the number of elements in the class grows like $N^{n(\Gamma)}(\alpha N)^{|\Gamma|/2}$. But $|\Gamma| \geqslant 2n(\Gamma)+2$, so the contribution of each class goes like N^{-1}. Then,

$$|\mathbb{R}_{k,n}(N)| \leqslant \frac{a(n,k)}{N} \tag{4.21$'$}$$

The other points of Lemma 4.1 are shown in essentially the same way as in Lemma 3.1 of ref. 1.

Now let us remember that

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) = \sum_{M,\Gamma} M(\Gamma)$$

where $M(\Gamma)$ is a "well-dressed" closed graph. Let us consider

$$\prod_{M,\gamma} [1 + M(\gamma)] = \sum_{\tilde{M},\Gamma} M(\Gamma) \tag{4.22}$$

where $M(\gamma)$ is a "well-dressed" loop and $\tilde{M}(\Gamma)$ is a closed graph which could also be "poorly dressed" (but always is a product of "well-dressed" loops). It is clear that, as in ref. 1, we can state the following result.

Lemma 4.2. We have

$$\prod_{M,\gamma} [1+M(\gamma)] \to \exp\left(v - \frac{1}{2}\langle v^2\rangle\right) \qquad \text{in distribution as } N\to\infty \tag{4.23}$$

where v is the same variable as in Lemma 4.1.

Let us now prove the lemma about the decaying of large graphs.

Lemma 4.3. For $\sqrt{\alpha}\,\beta J^2/(1-\beta J^2)<1$, the contribution to $\hat{\mathbb{Z}}_N$ of large graphs decays exponentially in the following sense:

$$\left\langle\!\!\left\langle \left[\sum_{M,\Gamma:|\Gamma|\geqslant k} M(\Gamma)\right]^2 \right\rangle\!\!\right\rangle \leqslant \text{const}\left[\frac{\alpha\beta^2 J^4}{(1-\beta J^2)^2}\right]^k \cdot \exp[(2k)^{1/2}] \tag{4.24}$$

Proof. The proof runs along the same lines as in Lemma 3.3 in ref. 1. Here we have only to consider that also if $M(\Gamma)\neq M'(\Gamma)$, the two marked graphs could not be orthogonal. $M(\Gamma)$ *and* $M'(\Gamma)$ are products of well-dressed loops, so it could happen that same of them are "equal" in the sense of Lemma 4.1 [i.e., elements of the second term in (4.13)]. So we can write

$$\begin{aligned}\text{LHS (4.24)} &= \sum_{M,M',\Gamma:|\Gamma|\geqslant k} \langle\!\langle M(\Gamma)\,M'(\Gamma)\rangle\!\rangle \\ &\leqslant \exp(-\varepsilon k) \prod_{M_1,M_2;\gamma} [1+\langle\!\langle M_1(\gamma)\,M_2(\gamma)\rangle\!\rangle \exp(\varepsilon\,|\gamma|)]\end{aligned} \tag{4.25}$$

So with

$$\exp(\varepsilon)\cdot\frac{\alpha(\beta J^2)^2}{(1-\beta J^2)^2} = 1 - \frac{1}{(2k)^{1/2}}$$

and by taking into account of the results of Lemma 4.1, it is easy to get the proof along the same lines as in ref. 1.

As we claimed, our approach reduces the model to a sort of "renormalized" SK model (at least in the region where this renormalization works, i.e., in the "ergodic" region), so especially from now on the lines of our proofs are very close to those of ref. 1. We report the main results and only stress the differences. Then we can prove

$$\prod_{M,\gamma} [1+M(\gamma)] - \sum_{n=0}^{l_N} \sum_{\substack{(m,\gamma_1),\ldots,(M,\gamma_n)\\ M(\gamma_i)\neq M(\gamma_j),\,|\gamma_i|\leqslant k_N}} M(\gamma_1)\cdots M(\gamma_n) \to 0$$

$$\text{in distribution as } N\to\infty \tag{4.26}$$

provided that k_N, l_N diverge when N goes to infinity. Equation (4.24) is proved by considering

$$\phi(z) = \prod_{M,\gamma} [1 + zM(\gamma)]$$

using the remainder formula

$$\left| \phi(1) - \sum_{n=0}^{l_N} \frac{1}{n!} \phi^{(n)}(0) \right| \leqslant \frac{R}{R-1} \frac{1}{R^{l+1}} \sup_{z \in \mathbb{C}, |z| = R} |\phi(z)| \tag{4.27}$$

for a function which is analytic is a disk of radius $R > 1$, and the results of Lemma 4.1.

Also by the same arguments used to prove Lemma 4.1 it is possible to show

$$\left\| \hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) - \sum_{n=0}^{l_N} \sum_{\substack{(M,\gamma_1) \cdots (M,\gamma_n) \\ M(\gamma_i) \neq M(\gamma_j), |\gamma_i| \leqslant k_N}} M(\gamma_1) \cdots M(\gamma_n) \right\|_2^2$$

$$= \left\| \sum_{M, \Gamma: |\Gamma| \geqslant k_N l_N}^{*} M(\Gamma) - \sum_{M, \Gamma: |\Gamma| \leqslant l_N k_N}^{**} M(\Gamma) \right\|_2^2$$

$$\leqslant \frac{\hat{\mathbb{C}}(l_N k_N)}{N} + \mathbb{C}' \exp\left\{ -l_N k_N \ln \left[\frac{\alpha\beta^2 J^4}{(1-\beta J^2)^2} \right]^{-2} + (l_N k_N)^{1/2} \right\} \tag{4.28}$$

where the first sum in the RHS is the same as in (4.13), while the second one is over all poorly dressed graphs and $\hat{\mathbb{C}}$ is a finite quantity which is independent of N. Choosing l_N, k_N in such a way that $l_N k_N \to \infty$ for $N \to \infty$ while $\hat{\mathbb{C}} l_N k_N / N \to 0$, it follows that

$$\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) - \prod_{M,\gamma} [1 + m(\gamma)] \to 0 \qquad \text{in distribution } N \to \infty \tag{4.29}$$

So we have proved Proposition 2.2.

From Proposition 2.2 it follows that

$$\frac{1}{N} \ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) - Q(\beta) \to 0 \qquad \text{in distribution,} \quad \forall \frac{\alpha\beta^2 J^4}{(1-\beta J^2)^2} < 1 \tag{4.30}$$

where $Q(\beta)$ is defined in (2.20).

From (4.28) it is possible to prove point (iii) of Proposition 2.1 by the bounds

$$\frac{1}{N} \ln \langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\rangle \geqslant \ln 2 \qquad \forall \beta, \alpha \tag{4.31}$$

and, with $Q_N := (1/N)\ln\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha)\rangle\rangle$,

$$\mathbb{P}\text{rob}\left[\frac{1}{N}\ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \geqslant Q_N + \varepsilon\right] \leqslant \exp(-\varepsilon N) \qquad \forall\beta \tag{4.32}$$

From (4.20)–(4.32), we are able to deduce points (iii) and (iv) of Proposition 2.1. To prove point (ii) is harder.

5. MEAN FREE ENERGY AND AN ORDER PARAMETER

Let us consider

$$\frac{d}{d\beta}\frac{1}{N}\ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) = \frac{1}{N^2}\sum_{\mu}\sum_{i<j} J^\mu_{ij}\langle\sigma_i\sigma_j\rangle_\beta \tag{5.1}$$

where $\langle\cdot\rangle_\beta$ is the thermal expectation. To decouple the variables J^μ_{ij} and $\langle\sigma_i\sigma_j\rangle_\beta$ it is not sufficient to cut off the bond J^μ_{ij} from the Hamiltonian. Nevertheless, let us define

$$\mathbb{H}_0(i, j, \lambda) = -\frac{1}{N}\left(\sum_{\mu\neq\lambda}\sum_{h<k} J^\mu_{ij}\sigma_h\sigma_k + \sum_{\substack{h<k\\(h,k)\neq(i,j)}} J^\lambda_{hk}\sigma_h\sigma_k\right) \tag{5.2}$$

$\mathbb{H}_0(i, j, \lambda)$ does not contain the bond J^λ_{ij}. Let us denote by $\langle\cdot\rangle_{i,j,\lambda}$ the thermal expectation w.r.t. the Gibbs distribution given by the Hamiltonian (5.2). Then,

$$\begin{aligned}\langle\sigma_i\sigma_j\rangle_\beta &= \frac{\langle\sigma_i\sigma_j\rangle_{i,j,\lambda} + \tanh[(\beta/N)\,\xi^\lambda_i\xi^\lambda_j]}{1+\tanh[(\beta/N)\,\xi^\lambda_i\xi^\lambda_{\langle}]\,\langle\sigma_i\sigma_j\rangle_{i,j,\lambda}}\\ &= \langle\sigma_i\sigma_j\rangle_{i,j,\lambda} + \tanh\left(\frac{\beta}{N}\xi^\lambda_i\xi^\lambda_j\right)(1-\langle\sigma_i\sigma_j\rangle^2_{i,j,\lambda}) + \mathbb{R}^0_N \end{aligned}\tag{5.3}$$

with $\mathbb{P}\text{rob}(|\mathbb{R}^0_N| > N^{-2+\delta}) \leqslant c''N^{-m\delta}$ for some $c''>0$ and $\forall m\in\mathbb{Z}_+$. Let us stress again that the variables J^λ_{ij} and $\langle\sigma_i\sigma_j\rangle_{(ij\lambda)}$ are not independent. We are able to control their dependence only at high temperature, namely in the ergodic region. To do that, let us write $\langle\sigma_i\sigma_j\rangle_{(ij\lambda)}$ as

$$\begin{aligned}\Bigg\{\Bigg(\mathbb{Z}^{(1)}_N(\underline{\xi}, \beta, \alpha)\sum_{\underline{\sigma}}\sigma_i\sigma_j\prod_\mu\Bigg[\sum_{\Gamma^c}^{*}\sigma_{\partial\Gamma^c}\mathbb{W}_\mu(\Gamma^c)\\ +\Bigg[\sum_{\Gamma:\partial\Gamma=\varnothing}\mathbb{W}_\mu(\Gamma)\Bigg]^{-1}\mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha)\Bigg]\Bigg)\Bigg/\\ (\mathbb{Z}^{(1)}_N(\underline{\xi}, \beta, \alpha)[\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) + \mathbb{R}^{(1)}_N])\Bigg\}_{(i,j,\lambda)}\end{aligned}\tag{5.4}$$

where the subscript (i, j, λ) denotes the absence of the bond J_{ij}^{λ}. Considering (2.17′), we can write

$$\langle \sigma_i \sigma_j \rangle_{(ij\lambda)} = \left\{ \frac{\sum_{\underline{\sigma}} \sigma_i \sigma_j \prod_\mu [\sum_{\partial \Gamma} \sigma_{\partial \Gamma} \sum^*_{\Gamma^c : \partial \Gamma^c = \partial \Gamma} \mathbb{W}_\mu(\Gamma^c) + \mathbb{R}_N^{(2)}}{\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha) + \mathbb{R}_N^{(1)}} \right\}_{(i,j,\lambda)} \tag{5.5}$$

with

$$\mathbb{P}\mathrm{rob}(|\mathbb{R}_N^{(2)}| > N^{-3/2+\delta}) \leqslant c_2 N^{-m\delta} \qquad \forall m \in \mathbb{Z}_+$$

By (4.8) we put

$$\langle \sigma_i \sigma_j \rangle_{(ij\lambda)} = \left\{ \frac{\sum_{\underline{\sigma}} \prod_\mu [\sum_{\tilde{\Gamma}} \mathbb{A}^{*\mu}(\tilde{\Gamma}) \sigma_{\tilde{\Gamma}}] \sigma_i \sigma_j + \mathbb{R}_N^{(2)}}{\sum_{\underline{\sigma}} \prod_\mu [\sum_{\tilde{\Gamma}} \mathbb{A}^{*\mu}(\tilde{\Gamma}) \sigma_{\tilde{\Gamma}}] + \mathbb{R}_N^{(1)}} \right\} \tag{5.6}$$

where

$$\mathrm{A}^{*\mu}(\tilde{\Gamma}) = \prod_{(h,k) \in \tilde{\Gamma}} \mathbb{A}^{*\mu}_{h,k}$$

$$\mathbb{A}^{*\mu}_{hk} = \begin{cases} \dfrac{\mathbb{W}_\mu(J^\mu_{hk})}{1 - \beta J^2} & \text{if} \quad (h, k, \mu) \neq (i, j, \lambda) \\[2ex] \beta J^2 \dfrac{\mathbb{W}_\mu(J^\mu_{hk})}{1 - \beta J^2} & \text{if} \quad (h, k, \mu) = (i, j, \lambda) \end{cases}$$

By performing the summation over $\underline{\sigma}$ in (5.6), we get

$$\langle \sigma_i \sigma_j \rangle_{(ij\lambda)} = \left\{ \frac{[\sum^*_{M^*, \Gamma} M^*(\Gamma) \cup \{i, j\}] + \mathbb{R}_N^{(2)}}{\sum^*_{M^*, \Gamma} M^*(\Gamma) + \mathbb{R}_N^{(1)}} \right\} \tag{5.7}$$

In (5.7) the star indicates that the bond J_{ij}^{λ} is absent and the upper summation is over all the well-dressed graphs containing one nude link (i, j) (nude, i.e., without the coefficient $[\beta J^2/(1 - \beta J^2)] \operatorname{tgh}[(\beta/N) J_{ij}^{\mu}]$). We can also say that $M^*(\Gamma) \cup \{i, j\}$ is a well-dressed graph containing the vertices (i, j), while i, j belong to an odd number of edges (only i, j!). Now $M^*(\Gamma) \cup \{i, j\}$ can be written as the product of a string with endpoints (i, j) for a simple closed graph, suitably dressed: $M^*(\Gamma_{ij}^c \circ \Gamma')$. But as usual

$$M^*(\Gamma) \cup \{i, j\} \neq M^*(\Gamma_{ij}^c) \circ M^*(\Gamma')$$

because $M^*(\Gamma_{ij}^c) \circ M^*(\Gamma')$ could be a graph that is not well dressed. We write

$$M^*(\Gamma) \cup \{i, j\} = \sum^*_{M^*, \Gamma_{ij}^c} M^*(\Gamma_{ij}^c) \sum_{(M^*, \Gamma' : \partial \Gamma' = \varnothing)} M^*(\Gamma') + \mathbb{R}_N^{(3)} \tag{5.8}$$

and $\mathbb{R}_N^{(3)}$ contains all graphs like $M^*(\Gamma) \cup \{i, j\}$ under the condition that the vertices i and/or j belong to at least two edges with the same index μ. This means that each graph contains an extra $1/N$ factor. Moreover, if we remain in the ergodic region, we know that for large N, $\hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)$ will remain far away from zero, let us say more than $N^{-\delta}$, with a probability larger than $1 - N^{-\delta/4\langle v^2\rangle^{1/2}}$. It follows that

$$\langle \sigma_i \sigma_j \rangle_{(ij\lambda)} = \sum_{M^*, \Gamma_{ij}^c}^{*} M^*(\Gamma_{ij}^c) + \mathbb{R}_N^{(4)} \tag{5.9}$$

and $\mathbb{R}_N^{(4)}$ is essentially

$$\frac{\mathbb{R}_N^{(3)}}{\sum^*_{M^*, \Gamma: \partial\Gamma = \varnothing} M^*(\Gamma)}$$

with

$$\mathbb{P}\mathrm{rob}\{|J_{ij}^{\lambda} \mathbb{R}_N^{(4)}| > N^{-(3/2 - \delta)}\} \leqslant N^{-1/2} \tag{5.9'}$$

Then

$$\langle\langle J_{ij}^{\lambda} \langle \sigma_i \sigma_j \rangle_{(ij\lambda)} \rangle\rangle \approx \frac{\beta^2 J^6}{N(1 - \beta J^2)} \tag{5.10}$$

because among the strings $M^*(\Gamma_{ij}^c)$ only the one edge string gives a contribution. In (5.10), $\approx$ means that we neglect contributions vanishing with N at least like $1/N^{1+\delta}$ for some $\delta > 0$. So we obtain

$$\frac{d}{d\beta} \left\{ \frac{1}{N} \ln \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) - \mathbb{Q}(\beta) \right\} \approx -\frac{1}{N^2} \sum_{\mu} \sum_{i<j} \left\langle\left\langle \frac{\beta}{N} (J_{ij}^{\mu})^2 \langle \sigma_i \sigma_j \rangle_{(ij\mu)}^2 \right\rangle\right\rangle \tag{5.11}$$

But with the previous estimates it is not hard to see that

$$\frac{1}{N^2} \sum_{\mu} \sum_{i<j} \left\langle\left\langle \frac{\beta}{N} (J_{ij}^{\mu})^2 \langle \sigma_i \sigma_j \rangle_{(ij\mu)}^2 \right\rangle\right\rangle = \frac{\alpha\beta J^4}{2} \langle\langle \tau(\underline{\xi}, \beta, \alpha)_N \rangle\rangle + o(N^{-1/2}) \tag{5.12}$$

By direct computations we are able to conclude that

$$\lim_N \langle\langle \tau(\underline{\xi}, \beta, \alpha)_N \rangle\rangle = 0 \tag{5.13}$$

At least in the ergodic region, conditions (ii)–(iv) of Proposition 2.1 remain equivalent in the Hopfield model.

APPENDIX

To prove (2.17), let us introduce the short-hand notations

$$\hat{\mathbb{Z}}_N(\mu, \underline{\sigma}) = \sum_{\partial\Gamma} \sigma_{\partial\Gamma} \sum_{\Gamma_C:\partial\Gamma_C=\partial\Gamma} \mathbb{W}_\mu(\Gamma) := \sum_{\Gamma}^{*} \sigma_{\partial\Gamma} \mathbb{W}_\mu(\Gamma) \tag{A1}$$

$$\mathbb{G}(\underline{\sigma}, \mu) := \mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha) \tag{A2}$$

$$\mathbb{Z}_N^{(1)}(\mu) := \sum_{\Gamma:\partial\Gamma=\varnothing} \mathbb{W}_\mu(\Gamma) \tag{A3}$$

where

$$\mathbb{G}_\mu(\underline{\sigma}, \underline{\xi}, \beta, \alpha) = \sum_{\Gamma}^{**} \sigma_{\partial\Gamma} \nu(\Gamma)\, \mathbb{W}_\mu(\Gamma) \tag{A2'}$$

In (A1) and (A2′) the summations are respectively over all graphs that are products of strings with at most one n.s.v., and over all the bad graphs. Then,

$$\begin{aligned}\mathbb{Z}_N(\underline{\xi}, \beta, \alpha) &= 2^{-N} \sum_{\underline{\sigma}} \prod_{\mu} [\mathbb{Z}_N^{(1)}(\mu)\, \hat{\mathbb{Z}}_N(\mu, \underline{\sigma}) + \mathbb{G}(\underline{\sigma}, \mu)] \\ &= 2^{-N} \sum_{\underline{\sigma}} \sum_{\chi\subset\{1,\dots,p\}} \prod_{j\in\chi} \mathbb{G}(\underline{\sigma}, j) \prod_{j\notin\chi} \mathbb{Z}_N^{(1)}(j)\, \hat{\mathbb{Z}}_N(j, \underline{\sigma})\end{aligned} \tag{A4}$$

where the last sum is over subsets of $\{1,\dots, p\}$, and p as usual is αN. We have

$$\begin{aligned}\text{RHS (A4)} = \sum_{\chi\subset\{1,\dots,p\}} \prod_{j\notin\chi} \mathbb{Z}_N^{(1)}(j) \sum_{\Gamma_1,\dots,\Gamma_p}^{\{\chi\}} \theta(\Gamma_1)\, \mathbb{W}_1(\Gamma_1) \\ \cdots \theta(\Gamma_p)\, \mathbb{W}_p(\Gamma_p) \cdot 2^{-N} \sum_{\underline{\sigma}} \sigma_{\partial\Gamma_1} \cdots \sigma_{\partial\Gamma_p}\end{aligned} \tag{A5}$$

where $\sum^{\{\chi\}}_{\Gamma_1,\dots,\Gamma_p}$ means that if $i\in\chi$, Γ_i is a "bad graph"; if $i\notin\chi$, Γ_i has no closed paths and not more than one n.s.v.; $\theta(\Gamma_i) = \nu(\Gamma_i)$ if $i\in\chi$, otherwise $\theta(\Gamma_i) = 1$. Of course, if $\chi = \varnothing$, we get

$$\sum_{\underline{\sigma}} \prod_{\mu} \mathbb{Z}_N^{(1)}(\mu)\, \hat{\mathbb{Z}}_N(\mu, \underline{\sigma}) = \mathbb{Z}_N^{(1)}(\underline{\xi}, \beta, \alpha)\, \hat{\mathbb{Z}}_N(\underline{\xi}, \beta, \alpha)$$

Let us consider

$$\begin{aligned}\mathbb{R}_N(\underline{\xi}, \beta, \alpha) = \sum_{\chi\subset\{1,\dots,p\}:\chi\neq\varnothing} \prod_{j\notin\chi} \mathbb{Z}_N^{(1)}(j) \sum_{\Gamma_1,\dots,\Gamma_p}^{\{\chi\}} \theta(\Gamma_1)\, \mathbb{W}_1(\Gamma_1) \\ \cdots \theta(\Gamma_p)\, \mathbb{W}_p(\Gamma_p) \cdot 2^{-N} \sum_{\underline{\sigma}} \sigma_{\partial\Gamma_1} \cdots \sigma_{\partial\Gamma_p}\end{aligned} \tag{A5'}$$

But $\sum_{\underline{\sigma}} \sigma_{\partial\Gamma_1} \cdots \sigma_{\partial\Gamma_p} = 0$ if $\partial(\Gamma_1 \circ \cdots \circ \Gamma_p) \neq \varnothing$. It follows that

$$\mathbb{R}_N(\underline{\xi}, \beta, \alpha) = \sum_{\substack{\chi \subset \{1,\ldots,p\} \\ \chi \neq \varnothing}} \prod_{j \notin \chi} \mathbb{Z}_N^{(1)}(j) \sum_{\substack{\Gamma_1,\ldots,\Gamma_p \\ \partial(\Gamma_1 \circ \cdots \circ \Gamma_p) = \varnothing}}^{\{x\}} \theta(\Gamma_1)\, \mathbb{W}_1(\Gamma_1) \cdots \theta(\Gamma_p)\, \mathbb{W}_p(\Gamma_p) \tag{A6}$$

Let us now prove (2.17) with $m = 1$:

$$\begin{aligned} &\mathbb{P}\text{rob}\left[\left|\frac{\mathbb{R}_N(\underline{\xi}, \beta, \alpha)}{\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha)\rangle\rangle}\right| > N^{-3/2+\delta}\right] \\ &\leqslant \frac{N^{3-2\delta}}{\langle\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha)\rangle\rangle^2} \\ &\quad \times \sum_{\substack{\chi,\chi':\chi,\chi' \neq \varnothing \\ \chi,\chi' \subset \{1\cdots p\}}} \sum_{\substack{\Gamma_1,\ldots,\Gamma_p \\ \partial(\Gamma_1 \circ \cdots \circ \Gamma_p) = \varnothing}}^{\{x\}} \sum_{\substack{\Gamma'_1,\ldots,\Gamma'_p \\ \partial(\Gamma'_1 \circ \cdots \circ \Gamma'_p) = \varnothing}}^{\{x'\}} \theta(\Gamma_1) \cdots \theta(\Gamma_p) \\ &\quad \times \theta(\Gamma'_1) \cdots \theta(\Gamma'_p) \\ &\quad \times \left\langle\!\!\left\langle \prod_{i \notin \chi} \mathbb{Z}_N^{(1)}(i) \prod_{j \notin \chi'} \mathbb{Z}_N^{(1)}(j)\, \mathbb{W}_1(\Gamma_1) \cdots \mathbb{W}_p(\Gamma_p)\, \mathbb{W}_1(\Gamma'_1) \cdots \mathbb{W}_p(\Gamma'_p) \right\rangle\!\!\right\rangle \end{aligned} \tag{A7}$$

Let us denote in the following formulas by $\bar{\Gamma}$ the "bad graphs" an by Γ without the bar the "good graphs," i.e., the graphs containing not more than one n.s.v. and no closed paths. Then for the mean $\langle\langle \cdot \rangle\rangle$ appearing after the sums in (A7),

$$\begin{aligned} \langle\langle \cdot \rangle\rangle = &\prod_{i \notin \{\chi \cup \chi'\}} \langle\langle \mathbb{Z}_N^{(1)}(i)^2\, \mathbb{W}_i(\Gamma_i)\, \mathbb{W}_i(\Gamma'_i)\rangle\rangle \\ &\times \prod_{i \in \{\chi \cap (\chi')^c\}} \nu(\bar{\Gamma}_i) \langle\langle \mathbb{Z}_N^{(1)}(i) \cdot \mathbb{W}_i(\bar{\Gamma}_i)\, \mathbb{W}_i(\Gamma'_i)\rangle\rangle \\ &\times \prod_{i \in \{(\chi)^c \cap \chi'\}} \nu(\bar{\Gamma}'_i) \langle\langle \mathbb{Z}_N^{(1)}(i)\, \mathbb{W}_i(\Gamma_i)\, \mathbb{W}_i(\bar{\Gamma}'_i)\rangle\rangle \\ &\times \prod_{i \in \{\chi \cap \chi'\}} \nu(\bar{\Gamma}_i)\, \nu(\bar{\Gamma}'_i) \langle\langle \mathbb{W}_i(\bar{\Gamma}_i)\, \mathbb{W}_i(\bar{\Gamma}'_i)\rangle\rangle \end{aligned} \tag{A8}$$

where $(\chi)^c = \{1 \cdots p\}/\{\chi\}$. Now

$$\mathbb{Z}_N^{(1)}(i) = \sum_{\Gamma:\partial\Gamma = \varnothing} \mathbb{W}_i(\Gamma)$$

From (A8) it is clear that, besides the constraints $\partial(\Gamma_1 \circ \cdots \circ \Gamma_p) = \varnothing$ and $\partial(\Gamma'_1 \circ \cdots \circ \Gamma'_p) = \varnothing$, we also have

$$\begin{aligned} &\partial(\Gamma_i \circ \Gamma'_i) = \varnothing, \quad \forall i \notin \chi \cup \chi', \qquad &\partial(\Gamma_i \circ \bar{\Gamma}'_i) = \varnothing, \quad \forall i \in \chi^c \cap \chi' \\ &\partial(\bar{\Gamma}_i \circ \Gamma'_i) = \varnothing, \quad \forall i \in \chi \cap \chi'^c, \qquad &\partial(\bar{\Gamma}_i \circ \bar{\Gamma}'_i) = \varnothing, \quad \forall i \in \chi \cap \chi' \end{aligned} \tag{A9}$$

Let us have a look at the most dangerous term: the first expectation $\langle\!\langle \cdot \rangle\!\rangle$ in (A8). The multigraph $\Gamma_i \circ \Gamma'_i$ has to be a loop or a product of loops, but not only! At least two vertices of Γ_i and of Γ'_i have to belong to some other graph Γ_j or $\bar{\Gamma}_j$, Γ'_j, or $\bar{\Gamma}'_j$, respectively. This fact gives an extra $1/N$ factor in each term appearing in the products of (A8). To better understand the situation, let us consider the case when $\Gamma_i \circ \Gamma'_i$ is a loop with k vertices. Then the contribution of all such loops goes like

$$N^{\#(\text{vertices})} \cdot \left(\frac{\beta J^2}{N}\right)^{\#(\text{edges})}$$

But the number of the "free" vertices is not more than $k-1$, while the number of the edges is k. To get an estimate, let us remove the constraints $\partial(\Gamma_1 \circ \cdots \circ \Gamma_p) = \varnothing$ and $\partial(\Gamma'_1 \circ \cdots \circ \Gamma'_p) = \varnothing$ and put instead a factor $1/N$ in front to each term in (A8). Let us remark that

$$\langle\!\langle \mathbb{Z}_N^{(1)}(i)^2 \rangle\!\rangle \leqslant \langle\!\langle \mathbb{Z}_N^{(1)}(i) \rangle\!\rangle^2 \left(1 + \frac{c_1}{N}\right)$$

Then we get

$$\sum_{\Gamma_i, \Gamma'_i} \langle\!\langle \mathbb{Z}_N^{(1)}(i)^2 \, \mathbb{W}_i(\Gamma_i) \, \mathbb{W}_i(\Gamma'_i) \rangle\!\rangle \leqslant \langle\!\langle \mathbb{Z}_N^{(1)}(i) \rangle\!\rangle^2 \left[1 + \frac{\langle\!\langle \mathbb{Z}_N^{(1)}(i) \rangle\!\rangle^2 - 1}{N}\right]$$

$$\sum_{\Gamma_i, \bar{\Gamma}'_i} v(\bar{\Gamma}'_i) \langle\!\langle \mathbb{Z}^{(1)}(i) \, \mathbb{W}_i(\Gamma_i) \, \mathbb{W}_i(\bar{\Gamma}'_i) \rangle\!\rangle \leqslant \frac{\langle\!\langle \mathbb{Z}^{(1)}(i) \rangle\!\rangle^3}{N^{5/2}}$$

$$\sum_{\bar{\Gamma}_i, \bar{\Gamma}'_i} v(\bar{\Gamma}_i) \, v(\bar{\Gamma}'_i) \langle\!\langle \mathbb{W}_i(\bar{\Gamma}_i) \, \mathbb{W}_i(\bar{\Gamma}'_i) \rangle\!\rangle \leqslant \frac{\langle\!\langle \mathbb{Z}_N^{(1)}(i) \rangle\!\rangle^2}{N^4} \tag{A10}$$

The estimates in (A10) are not the best ones, but they are sufficient for our purposes. Then, denoting by $\mathbb{Z}$ the expectation $\langle\!\langle \mathbb{Z}_N^{(1)}(i) \rangle\!\rangle$, we get

$$\text{LHS (A7)} \leqslant \frac{N^{3-2\delta}}{\langle\!\langle \mathbb{Z}_N(\underline{\xi}, \beta, \alpha) \rangle\!\rangle^2} \sum_{k=1}^{p} \sum_{k''=0}^{k} \binom{p}{k} \binom{k}{k''} \sum_{k'=1}^{p-k} \binom{p-k}{k'} \times \left[Z^2 \left(1 + \frac{Z^2 - 1}{N}\right)\right]^{p-k-k'} \left(\frac{Z^3}{N^{5/2}}\right)^{k+k'-k''} \left(\frac{Z^2}{N^4}\right)^{k''} \tag{A11}$$

By performing the computations, we arrive at

$$\text{LHS (A7)} \leqslant c_1 N^{-2\delta} \exp[\alpha \langle\!\langle \mathbb{Z}^{(1)}(i) \rangle\!\rangle^2 - 1] \tag{A12}$$

Taking into account point (ii) of Proposition 2.1, it is easy to prove in the same way (2.17″).

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